A New Approach to Determinants

Isamu YAMAZAKI

Information Science Course, Department of Information Science and Electronics, Tsukuba College of Technology

Abstract: A new approach to determinants is proposed. Compared to the standard approach, which requires students to understand permutations and transpositions, the new approach adopts more intuitive ideas named *n-hisha selection* and *right-upward relation*. The *n-hisha selection* is a selection of *n* positions from *n* by *n* matrix under the condition that every selected position is alone in its row and its column. By this approach, the principal properties of determinants can be derived relatively easily. For example, the fact that "the determinant of a transposed matrix is equal to that of the original matrix," is the direct consequence of this definition, because the definition itself is symmetrical regarding its diagonal line. Adopting this approach in lectures of elementary mathematics for freshmen of Tsukuba College of Technology resulted in saving two lecture periods and two accompanying exercise periods, or saving 320 minutes in total.

**Key Words:** definition of determinants, linear algebra, n-hisha selection.

1. Introduction

Linear algebra is the most fundamental component in mathematics. It presents powerful tools in a wide variety of areas from theoretical science to engineering, including computer science. So, every student aiming to become a scientist or an engineer, is expected to learn and understand linear algebra.

Linear algebra includes various interesting and fruitful topics, such as *linear space*, *linear dependencies*, *subspace*, *projection*, *linear transformation*, *eigenvalue*, and *eigenvector*. Before discussing those topics, however, students must become familiar with the elementary concepts: *vector*, *matrix*, and *determinant*.

Compared with *vector* or *matrix*, *determinant* is somewhat difficult to learn.

The difficulty arises from the point that the traditional definition of determinants utilizes the concepts of permutation and transposition. The properties of determinants are deduced from the very properties of those concepts. So, students are usually forced to make a several-hour detour to learn those concepts before learning the *determinant* itself.

In this paper, the author proposes a more straightforward introduction to

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1 In this paper, the word *transpose* or *transposition* is used with two different meanings. When discussing permutation (i.e., a sequence of elements), *transposition* means interchange of the elements between two positions in a permutation. When talking about matrix, *transposition* means to turn or to *reverse* a matrix with respect to its diagonal line.
determinants, never using permutation nor transposition, and still maintaining mathematical strictness.

2. A brief survey of the standard approach

2.1 The standard definition

According to the standard approach, determinant is defined as follows.

Let $M$ be the set of all possible permutations generated from $(1\ 2\ \cdots\ n)$. Then the standard definition of determinants is as follows.

$$\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = \sum_{(p_1\ p_2\ \cdots\ p_n)\in M} \text{sign}(p_1\ p_2\ \cdots\ p_n)a_{1p_1}a_{2p_2}\cdots a_{np_n},$$

where $\text{sign}(p)$ is the parity of the permutation $p=(p_1\ p_2\ \cdots\ p_n)$ defined as

$$\text{sign}(p) = \begin{cases} 
1 & \text{if } p \text{ is even permutation}, \\
-1 & \text{if } p \text{ is odd permutation}.
\end{cases}$$

An even (odd) permutation is the permutation that can be converted into the prime permutation $(1\ 2\ 3\ \cdots\ n)$ by even (odd) time applications of transpositions. A transposition is an operation to interchange the elements between two positions in a permutation. Any permutation can be converted to any other permutation by successively applying proper transpositions.

This definition of parity is meaningless if the number of times can be both odd and even depending on the sequence of transpositions adopted for the same permutation. Therefore, before introducing the above definition, it is necessary to prove that such a situation cannot arise.

As a result, the standard approach to determinants usually has the preliminary discussions including the following contents.

(p1) introduce permutations.
(p2) introduce transpositions.
(p3) show the fact that any permutation can be converted to the prime permutation by successively applying proper transpositions.
(p4) show the fact that, given a permutation, the number of times of transpositions applied for converting the permutation to the prime one is definitely even or odd, not depending on the sequence adopted.
(p5) define the parity of permutation.

These preliminary discussions take two lecture periods out of a total of six lecture periods related to determinants.

2.2 The derivation of the properties of determinants

After having defined determinants, the standard approach proceeds to the next step to show the properties of determinants. Among those properties, next three properties depend on the definition of determinants in showing their correctness.

(1) “The determinant of a matrix is equal to that of the transposed matrix,” or
where $A^T$ denotes the transposed matrix of $A$. In the standard definition, this means
\[
|A| = |A^T|,
\]
where $A^T$ denotes the transposed matrix of $A$. In the standard definition, this means
\[
\sum_{(p_1 p_2 \cdots p_n) \in M} \text{sign}(p_1 p_2 \cdots p_n) a_{1p_1} a_{2p_2} \cdots a_{np_n} = \sum_{(p_1 p_2 \cdots p_n) \in M} \text{sign}(p_1 p_2 \cdots p_n) a_{1p_1} a_{2p_2} \cdots a_{np_n}.
\]
The RHS (righthand side) of this equality can be rewritten as
\[
\sum_{(p_1 p_2 \cdots p_n) \in M} \text{sign}(p_1 p_2 \cdots p_n) a_{1q_1} a_{2q_2} \cdots a_{nq_n},
\]
where $a_{1q_1} a_{2q_2} \cdots a_{nq_n}$ is the re-arrangement of $a_{1p_1} a_{2p_2} \cdots a_{np_n}$. Therefore in order to show the above equality, it is sufficient to say that, if $a_{1q_1} a_{2q_2} \cdots a_{nq_n}$ is a re-arrangement of $a_{1p_1} a_{2p_2} \cdots a_{np_n},$
\[
\text{sign}(q_1 q_2 \cdots q_n) = \text{sign}(p_1 p_2 \cdots p_n).
\]
This requires some sophisticated discussion.

(2) “Interchange of two columns (or two rows) changes the sign of the determinant,” or
\[
\begin{vmatrix}
  a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\
  a_{21} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2n} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn}
\end{vmatrix}
\]
\[
= -
\begin{vmatrix}
  a_{11} & \cdots & a_{1j} & \cdots & a_{1i} & \cdots & a_{1n} \\
  a_{21} & \cdots & a_{2j} & \cdots & a_{2i} & \cdots & a_{2n} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{n1} & \cdots & a_{nj} & \cdots & a_{ni} & \cdots & a_{nn}
\end{vmatrix}
\]
In the standard approach, this is the straightforward conclusion of the definition of the parity of permutation. In other words, just to show this property, the parity of permutation is defined through the number of times of transpositions applied for converting the permutation to the prime one.

(3) “Extracting rule,” or
\[
\begin{vmatrix}
  a_{11} & 0 & 0 & \cdots & 0 \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{vmatrix}
= a_{11}
\begin{vmatrix}
  a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots & \vdots & \vdots & \vdots \\
  a_{n2} & a_{n3} & \cdots & a_{nn}
\end{vmatrix}
\]
This can be shown as follows. First, in the definition of the lefthand side of the above equation, $a_{im} = 0$ except the case $p_i = 1$, therefore
\[
\text{LHS} = \sum_{(p_1 p_2 \cdots p_n) \in M'} \text{sign}(p_1 p_2 \cdots p_n) a_{1p_1} a_{2p_2} \cdots a_{np_n} \\
= a_{11} \sum_{(p_2 \cdots p_n) \in M'} \text{sign}(1 p_2 \cdots p_n) a_{2p_2} \cdots a_{np_n},
\]

where \( M' \) denotes the set of whole permutations generated from \((2 3 \cdots n)\).

Second, \((1 p_2 \cdots p_n) = \text{sign}(p_2 \cdots p_n)\) because the sequence of transpositions which converts \((1 p_2 p_3 \cdots p_n)\) into \((1 2 3 \cdots n)\) also converts \((p_2 p_3 \cdots p_n)\) into \((2 3 \cdots n)\). Therefore

\[
a_{11} \sum_{(p_2 \cdots p_n) \in M'} \text{sign}(1 p_2 \cdots p_n) a_{2p_2} \cdots a_{np_n} \\
= a_{11} \sum_{(p_2 \cdots p_n) \in M'} \text{sign}(p_2 \cdots p_n) a_{2p_2} \cdots a_{np_n} = \text{RHS}.
\]

2.3 Consideration on the standard approach

After learning determinants through those standard definitions, engineers and scientists begin to use determinants in their work: calculating them, or using the properties of determinants. But it is almost unnecessary for them in the rest of their lives to remember how to derive the properties. Most students consider the derivation only once just when they learn determinants at universities. So, it is preferable to make the derivation as simple as possible, for example, by avoiding the subjects as permutations or transpositions in the introduction to determinants, unless the study further progresses to group theory, in which permutation group is a good example.

2.4 Modified standard definition

Before proceeding to the new approach, it will be adequate here to point out that the parity of the permutation can be defined using the number of inverted pairs in the permutation: the parity is 1 if the number is even, and \(-1\) if odd. For example, the parity of the permutation \((1 2 5 4 3)\) is \(-1\), because it has three inverted pairs: \((5, 4)\), \((5, 3)\), and \((4, 3)\). Let the definition of determinants which adopts this definition of parity be referred to as the modified standard definition of determinants.

3. The new approach

3.1 Key idea of the new approach

Here, let us introduce a more intuitive definition of determinants without using permutation nor transposition. Instead, the next two concepts play important roles in the new definition.

(a) \(n\)-hisha selections.

\(N\)-hisha selection is a selection of \(n\) positions from \(n\) by \(n\) matrix in which every row and column has exactly one selected position. For example, \(P_1\) and \(P_2\) in the next figure are \(n\)-hisha selections. (The mark “*” shows the selected position.)
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Hisha is a piece used in shogi (Japanese chess). On the shogi board, hisha is allowed to reach any position on the row or the column on which it is located. So an \( n \)-hisha selection can be seen as a solution of the game that requires the player to place \( n \) pieces of hisha on an \( n \) by \( n \) square board to satisfy the condition that any hisha cannot reach another’s position. Though this game is much easier than the famous Eight Queen puzzle, the name \( n \)-hisha selection comes from the analogy between them.

This concept is visual and easy to understand.

(b) right-upward relations and the parity of \( n \)-hisha selection.

The next step is to define the parity of the \( n \)-hisha selection. It is defined through the number of the right-upward relations among the selected positions in the \( n \)-hisha selection. If the number is even, the parity is 1, if odd, then \(-1\). For example, as shown in the next figure, \( P_1 \) has four right-upward relations, therefore the parity of \( P_1 \) is 1. And \( P_2 \) has seven right-upward relations, therefore the parity of \( P_2 \) is \(-1\).

This definition of parity is also visually conceivable.

3.2 The new definition

Let \( S \) be the set of all \( n \)-hisha selections. Then the new definition of the determinants is as follows.

\[
\begin{vmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = \sum_{P \in S} \epsilon(P) t(P),
\]

where \( \epsilon(P) \) denotes the parity of the \( n \)-hisha selection \( P \), and \( t(P) \) denotes the product term of the elements which exist at the selected positions in the \( P \).

There is a one to one correspondence between \( n \)-hisya selections and permutations.
And it can be shown that the number of right-upward relations in a n-hisya selection is exactly equal to the number of inversions in the corresponding permutation. So the determinant by the new definition exactly coincides with that of the modified standard definition.

3.3 The derivation of the properties of determinants

Now consider how to derive the properties of determinants under the new approach.

(1) “The determinant of a matrix is equal to that of the transposed matrix,” or

$$|A| = |A^T|.$$  

This is the direct consequence of the new definition, because the definition is symmetrical with respect to the diagonal line, i.e., for each n-hisha selection $P$ in $A$, $P^T$ is also n-hisha selection in $A^T$, and the right-upward relation in $P$ is still the right-upward relation in $P^T$.

(2) “Interchange of two columns (or two rows) changes the sign of the determinant.”

This can be shown by a two-step discussion. In the first step we limit the discussion to the case where the two columns (or rows) interchanged are adjacent. In this case the execution of the interchange increases or decreases the right-upward relations in every n-hisha selection by 1, which results in changing parity of the n-hisha selection, and therefore changing the sign of the determinant. In the second step, we discuss the interchange between non-adjacent columns (or rows), say, between $i$-th column and $j$-th column ($i < j$). In this case, the original interchange can be performed by successive interchanges of adjacent columns, the number of whose times is $2(j - i) - 1$, i.e., odd number, which again results in the change of the sign of the determinant.

(3) “Extracting property,” or

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}.$$  

This can be derived from the next consideration. Let $S$ and $S'$ be the sets of all n-hisha selections in the lefthand side determinant and the righthand side determinant, respectively, and let $S''$ be the subset of $S$ whose each element includes $a_{11}$ position. Then

(1) if $P \in S''$ then $t(P) = a_{11}$, $t(Q)$, otherwise $t(P) = 0$,

(2) if $P \in S''$ then $\epsilon(P) = \epsilon(Q)$ because $a_{11}$ does not contribute to any right-upward relation,
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where \( Q \) means the \( n \)-hisha selection of the righthand side determinant corresponding to \( P \) except \( a_{11} \) position, as shown in the next example.

\[
P:
\begin{vmatrix}
a_{11} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & a_{24} & \cdots & \cdots \\
\cdots & a_{32} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & a_{45} & \cdots \\
\cdots & \cdots & \cdots & \cdots & a_{53}
\end{vmatrix}
\]

\[
Q:
\begin{vmatrix}
\cdots & \cdots & a_{24} & \cdots & \cdots \\
\cdots & a_{32} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & a_{45} & \cdots \\
\cdots & \cdots & \cdots & \cdots & a_{53}
\end{vmatrix}
\]

Therefore,

\[
LHS = \sum_{P \in S} \varepsilon(P)t(P) = \sum_{P \in S''} \varepsilon(P)t(P) = \sum_{Q \in S'} \varepsilon(Q)t(Q) = a_{11} \sum_{Q \in S'} \varepsilon(Q)t(Q) = RHS.
\]

3.4 Practical applications to the lecture and the textbook

The new approach to determinants mentioned above was adopted at an introductory lecture to linear algebra for freshmen in the Information Science Course, Tsukuba College of Technology, since 2001. The positive effects of adopting this new approach were as follows.

(1) One student who learned determinants by this new approach reported that she had the impression that the determinant was easier to understand than the matrix or vector.

(2) Two lecture periods (and two accompanied exercise periods) have been released from the subjects permutations and transpositions. The released periods could be now dedicated, instead, to other topics in linear algebra, such as vector space, eigenvalue and eigenvector.

(3) The new approach was also effective in reducing the page count of the textbook for the introduction of determinants by approximately 3 pages.

On the other hand, one can argue that the new approach has some negative effects also. That is, from the viewpoint of the educational effect for mathematics, the new approach deprives students of training opportunities through understanding permutations and transpositions. But those subjects seem to be more appropriately dealt with in other areas, such as group theory, where permutation group is a good example.

4. Conclusion

In this paper an intuitive definition of determinants is proposed, which adopts new concepts named \( n \)-hisha selection and right-upward relation. This new definition has been originally developed for a lecture for the hearing impaired in TCT (Tsukuba College of Technology: a college for the hearing impaired and the visually impaired). But this method can be applied not only at TCT but also at any college or any university, because of its intuitive nature.

The author would like to say a few words on the relation between this new approach and auditory difficulties. The new definition of determinants is highly
dependent on visual images, such as n-hisha selections and right-upward relations. And this fact seems to help students with hearing impairment to understand them more easily, because they seem to have a tendency to prefer visual images to literal concepts, as is shown in one of the students' impressions on the determinant by the new definition.

References